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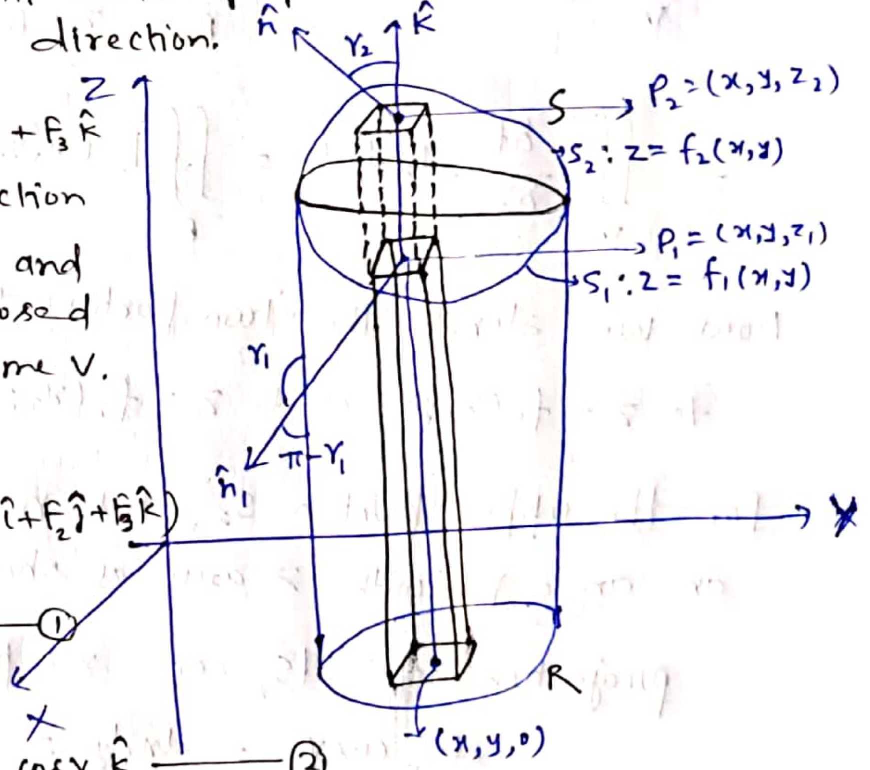
Gauss' divergence theorem: Gauss' divergence theorem provides a relation between surface integral and volume integral.

Statement:- If $\vec{F}$ is a vector point function which is continuous and differentiable over a closed surface S enclosing a volume V then surface integral of the vector point function $\vec{F}$ over the surface S is equal to volume integral of divergence of the vector point function $\vec{F}$ taken over the volume V.

i.e, $$\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \mathrm{div}\,\vec{F}\,dv$$

where $\hat{n}$ is a unit vector perpendicular to surface S in outward direction.

proof:- Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector point function which is continuous and differentiable over a closed surface S enclosing volume V.



Now $\mathrm{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\Rightarrow \mathrm{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{——— (1)}$$

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k} \quad \text{——— (2)}$$

where α, β and γ are angles made by $\hat{n}$ with x, y and z axes respectively.

Now $\vec{F}\cdot\hat{n} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$

$$\Rightarrow \vec{F}\cdot\hat{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma \quad \text{——— (3)}$$

Now we have $$\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\vec{F}\,dV$$

$$\Rightarrow \oiint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS = \iiint_{RV} \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dx\,dy\,dz \quad (4)$$

Using equ^n ① and ③ and $dV = dx\,dy\,dz$.

Let the closed surface S is such that any line parallel to any coordinate axis cuts the surface S in atmost two points.

Let R be orthogonal projection of the surface S on x-y plane. A line through a point $(x, y, 0)$ of R meets the boundary of S in two points. Let z coordinates of the two points be $z_1 = f_1(x,y)$ and $z_2 = f_2(x,y)$ when $z_2 \geq z_1$.

Now
$$\iiint_V \frac{\delta F_3}{\delta z}\, dx\,dy\,dz = \iint_R \left(\int_{z_1 = f_1(x,y)}^{z_2 = f_2(x,y)} \frac{\delta F_3}{\delta z}\, dz \right) dx.dy = \iint_R \Big[F_3(x,y,z) \Big]_{z_1 = f_1(x,y)}^{z_2 = f_2(x,y)} dx\,dy$$

$$\Rightarrow \iiint_V \frac{\delta f_3}{\delta z}.dx\,dy\,dz = \iint_R \left[F_3(x,y,f_2) - F_3(x,y,f_1) \right] dx\,dy \quad (5)$$

Now we denote the two portions of the surface S corresponding to $z = f_1(x,y)$ and $z = f_2(x,y)$ as S_1 and S_2 respectively.

For the upper portion S_2, let normal unit vector $\hat{n}_2$ makes an angle γ_2 with z axis as shown in fig. Here γ_2 = acute angle.

projection of dS_2 on xy plane is

$$dS_2 \cos\gamma_2 = dx\,dy = dS_2 \cdot \hat{n}_2 \cdot \hat{k} \quad (6)$$

For the lower portion S_2, Let the normal unit vector $\hat{n}_1$ makes an angle γ_1 with z axis as shown in fig. Here γ_1 = obtuse

projection dS_1 on xy plane is

$$dx\,dy = dS_1 . \cos(\pi - \gamma_1) \Rightarrow dx\,dy = -dS_1 \cos\gamma = -\hat{n}_1 \cdot \hat{k} \quad (7)$$

Using equ^n ⑥ and ⑦ in equ^n ⑤, we get

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$$\iiint_V \frac{\delta F_3}{\delta z}\,dx\,dy\,dz = \iint_R F_3(x,y,f_2)\,dx\,dy - \iint_R F_3(x,y,f_1)\,dx\,dy$$

$$= \iint_{S_2} F_3(x,y,f_2)\cos\gamma_2\,dS + \iint_{S_1} F_3(x,y,f_1)\cos\gamma_1\,dS$$

$$\iiint_V \frac{\delta F_3}{\delta z}\,dx\,dy\,dz = \oiint_S F_3\cos\gamma\,dS \quad \text{———— (8)}$$

Similarly

$$\iiint_V \frac{\delta F_2}{\delta y}\,dx\,dy\,dz = \oiint_S F_2\cos\beta\,dS \quad \text{———— (9)}$$

and $$\iiint_V \frac{\delta F_1}{\delta x}\,dx\,dy\,dz = \oiint_S F_1\cos\alpha\,dS \quad \text{———— (10)}$$

Adding eqn (8), (9) and (10), we get

$$\oiint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS = \iiint_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_1}{\delta y} + \frac{\delta F_1}{\delta z}\right)dx\,dy\,dz$$

$$\Rightarrow \boxed{\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV}$$ proved.